三角関数の積分

[1] 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{dx}{\sin x}$

【解答】

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\,dx$$

$$= \int \frac{\left(\tan\frac{x}{2}\right)'}{\tan\frac{x}{2}}\,dx = \boldsymbol{\log\left|\tan\frac{x}{2}\right| + C}$$

[別]

$\tan\frac{x}{2} = t$ とおくと $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\tan x = \frac{2t}{1-t^2}$

$\tan\frac{x}{2} = t$ より $\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{dx}{dt} = 1$

ここで, $\frac{1}{\cos^2\frac{x}{2}} = 1 + \tan^2\frac{x}{2} = 1 + t^2$ であるから $\frac{dx}{dt} = \frac{2}{1+t^2}$

$$\int \frac{dx}{\sin x} = \int \frac{1}{\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{dt}{t}$$

$$= \log|\,t\,| + C = \boldsymbol{\log\left|\tan\frac{x}{2}\right| + C}$$

(2) $\displaystyle\int \frac{dx}{\cos x}$

【解答】

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(x + \frac{\pi}{2}\right)}$$

$$= \log\left|\tan\left\{\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right\}\right| + C = \boldsymbol{\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C}$$

(3) $\displaystyle\int \frac{dx}{1+\cos x}$

【解答】

$$\int \frac{dx}{1+\cos x} = \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}} = \boldsymbol{\tan\frac{x}{2} + C}$$

(4) $\displaystyle\int \frac{dx}{1-\cos x}$

【解答】

$$\int \frac{dx}{1-\cos x} = \frac{1}{2}\int \frac{dx}{\sin^2\frac{x}{2}} = \boldsymbol{-\frac{1}{\tan\frac{x}{2}} + C}$$

(5) $\displaystyle\int \frac{dx}{1+\sin x}$

【解答】

$$\int \frac{dx}{1+\sin x} = \int \frac{dx}{1-\cos\left(x - \frac{\pi}{2}\right)}$$

$$= -\frac{1}{\tan\left\{\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right\}} + C = \boldsymbol{-\frac{1}{\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)} + C}$$

(6) $\displaystyle\int \frac{dx}{1-\sin x}$

【解答】

$$\int \frac{dx}{1-\sin x} = \int \frac{dx}{1+\cos\left(x - \frac{\pi}{2}\right)}$$

$$= \tan\left\{\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right\} + C = \boldsymbol{\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + C}$$

(7) $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

【解答】

$\tan\dfrac{x}{2}=t$ とおくと $\sin x=\dfrac{2t}{1+t^2},\ \cos x=\dfrac{1-t^2}{1+t^2},\ \tan x=\dfrac{2t}{1-t^2}$

$\tan\dfrac{x}{2}=t$ より $\dfrac{1}{2}\cdot\dfrac{1}{\cos^2\dfrac{x}{2}}\cdot\dfrac{dx}{dt}=1$

ここで, $\dfrac{1}{\cos^2\dfrac{x}{2}}=1+\tan^2\dfrac{x}{2}=1+t^2$ であるから $\dfrac{dx}{dt}=\dfrac{2}{1+t^2}$

$$\begin{aligned}\int\frac{dx}{1+\sin x+\cos x}&=\int\frac{1}{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt\\&=\int\frac{1}{\dfrac{2t+2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt\\&=\int\frac{dt}{t+1}=\log|\,t+1\,|+C=\boldsymbol{\log\left|\tan\frac{x}{2}+1\right|+C}\end{aligned}$$

(8) $I=\displaystyle\int\frac{\sin x}{1+\sin x+\cos x}\,dx,\ J=\int\frac{\cos x}{1+\sin x+\cos x}\,dx$

【解答】

$$\begin{aligned}I+J&=\int\frac{\sin x+\cos x}{1+\sin x+\cos x}\,dx=\int\left(1-\frac{1}{1+\sin x+\cos x}\right)dx\\&=x-\log\left|\tan\frac{x}{2}+1\right|+C_1\\J-I&=\int\frac{\cos x-\sin x}{1+\sin x+\cos x}\,dx=\int\frac{(1+\sin x+\cos x)'}{1+\sin x+\cos x}\,dx\\&=\log|\,1+\sin x+\cos x\,|+C_2\\I&=\frac{1}{2}\left(x-\log\left|\tan\frac{x}{2}+1\right|-\log|\,1+\sin x+\cos x\,|\right)\\&=\boldsymbol{\frac{1}{2}\left\{x-\log(1+\sin x)\right\}+C}\\J&=\frac{1}{2}\left(x-\log\left|\tan\frac{x}{2}+1\right|+\log|\,1+\sin x+\cos x\,|\right)\\&=\boldsymbol{\frac{1}{2}\left\{x-\log(1+\cos x)\right\}+C}\end{aligned}$$

(9) $\displaystyle\int\frac{\sin x}{1+\sin x+\cos x}\,dx$

【解答】

$\tan\dfrac{x}{2}=t$ とおくと $\sin\dfrac{x}{2}=\dfrac{t}{\sqrt{1+t^2}},\ \cos\dfrac{x}{2}=\dfrac{1}{\sqrt{1+t^2}}$

$\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}=\dfrac{2t}{1+t^2}$

$\cos x=2\cos^2\dfrac{x}{2}-1=\dfrac{1-t^2}{1+t^2}$

$\tan x=\dfrac{\sin x}{\cos x}=\dfrac{2t}{1-t^2}$

$\tan\dfrac{x}{2}=t$ より $\dfrac{1}{2}\cdot\dfrac{1}{\cos^2\dfrac{x}{2}}\cdot\dfrac{dx}{dt}=1$

ここで, $\dfrac{1}{\cos^2\dfrac{x}{2}}=1+\tan^2\dfrac{x}{2}=1+t^2$ であるから $\dfrac{dx}{dt}=\dfrac{2}{1+t^2}$

$$\begin{aligned}\int\frac{\sin x}{1+\sin x+\cos x}\,dx&=\int\frac{\dfrac{2t}{1+t^2}}{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt\\&=\int\frac{\dfrac{2t}{1+t^2}}{\dfrac{2t+2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt\\&=\int\frac{2t}{(t+1)(1+t^2)}\,dt=\int\left(-\frac{1}{t+1}+\frac{1}{2}\cdot\frac{2t}{t^2+1}+\frac{1}{t^2+1}\right)dt\\&=-\log|\,t+1\,|+\frac{1}{2}\log(t^2+1)+\arctan t+C\\&=-\log\left|\frac{t+1}{\sqrt{1+t^2}}\right|+\frac{x}{2}+C\\&=\boldsymbol{-\log\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|+\frac{x}{2}+C}\\&=\boldsymbol{\frac{1}{2}\left\{x-\log(1+\sin x)\right\}+C}\end{aligned}$$

(10) $I=\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx,\ J=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx$

【解答】

$x=\dfrac{\pi}{2}-t$ とおくと $dx=-dt$

x	$0 \to \frac{\pi}{2}$
t	$\frac{\pi}{2} \to 0$

$\sin x=\cos t,\ \cos x=\sin t$ より

$$J=\int_{\frac{\pi}{2}}^{0} \frac{\sin t}{\cos t+\sin t}\cdot(-1)\,dt=\int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t+\cos t}\,dt=I$$

別

$$J-I=\int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{\sin x+\cos x}\,dx=\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)'}{\sin x+\cos x}\,dx$$

$$=\Big[\log|\sin x+\cos x|\Big]_0^{\frac{\pi}{2}}=\log 1-\log 1=0$$

$$I+J=\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}} dx=\frac{\pi}{2}$$

$\therefore\ I=J=\boldsymbol{\dfrac{\pi}{4}}$

(11) $I=\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x+\cos x}\,dx,\ J=\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x+\cos x}\,dx$

【解答】

$x=\dfrac{\pi}{2}-t$ とおくと $dx=-dt$

x	$0 \to \frac{\pi}{2}$
t	$\frac{\pi}{2} \to 0$

$\sin x=\cos t,\ \cos x=\sin t$ より

$$J=\int_{\frac{\pi}{2}}^{0} \frac{\sin^3 t}{\cos t+\sin t}\cdot(-1)\,dt=\int_0^{\frac{\pi}{2}} \frac{\sin^3 t}{\sin t+\cos t}\,dt=I$$

$$I+J=\int_0^{\frac{\pi}{2}} \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)(\sin^2 x-\sin x\cos x+\cos^2 x)}{\sin x+\cos x}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \left(1-\frac{1}{2}\sin 2x\right)dx$$

$$=\left[x+\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}=\left(\frac{\pi}{2}-\frac{1}{4}\right)-\frac{1}{4}=\frac{\pi-1}{2}$$

$\therefore\ I=J=\boldsymbol{\dfrac{\pi-1}{4}}$

(12) $I=\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x+\cos x}\,dx,\ J=\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x}\,dx$

【解答】

$x=\dfrac{\pi}{2}-t$ とおくと $dx=-dt$

x	$0 \to \frac{\pi}{2}$
t	$\frac{\pi}{2} \to 0$

$\sin x=\cos t,\ \cos x=\sin t$ より

$$J=\int_{\frac{\pi}{2}}^{0} \frac{\sin t}{1+\cos t+\sin t}\cdot(-1)\,dt=\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+\sin t+\cos t}\,dt=I$$

別

$$J-I=\int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{1+\sin x+\cos x}\,dx=\int_0^{\frac{\pi}{2}} \frac{(1+\sin x+\cos x)'}{1+\sin x+\cos x}\,dx$$

$$=\Big[\log|1+\sin x+\cos x|\Big]_0^{\frac{\pi}{2}}=\log 2-\log 2=0$$

$$I+J=\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{1+\sin x+\cos x}\,dx=\int_0^{\frac{\pi}{2}} \left(1-\frac{1}{1+\sin x+\cos x}\right)dx$$

$$=\left[x-\log\left|\tan\frac{x}{2}+1\right|\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}-\log 2$$

$\therefore\ I=J=\boldsymbol{\dfrac{\pi-2\log 2}{4}}$