有理関数の積分 (部分分数分解)

1 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{dx}{(x+1)(x+2)^2}$

【解答】$\displaystyle\frac{1}{(x+1)(x+2)^2} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$ とおいて

$a=1,\ b=-1,\ c=-1$

$$\int \frac{dx}{(x+1)(x+2)^2} = \int \left\{ \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2} \right\} dx$$

$$= \log|x+1| - \log|x+2| + \frac{1}{x+2} + C = \log\left|\frac{x+1}{x+2}\right| + \frac{1}{x+2} + C$$

(2) $\displaystyle\int \frac{x+1}{x^3-1}\,dx$

【解答】$\displaystyle\frac{x+1}{x^3-1} = \frac{a}{x-1} + \frac{b\,x+c}{x^2+x+1}$ とおいて

$a=\dfrac{2}{3},\ b=-\dfrac{2}{3},\ c=-\dfrac{1}{3}$

$$\int \frac{x+1}{x^3-1}\,dx = \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx$$

$$= \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{(x^2+x+1)'}{x^2+x+1}dx$$

$$= \frac{2}{3}\log|x-1| - \frac{1}{3}\log(x^2+x+1) + C = \frac{1}{3}\log\frac{(x-1)^2}{x^2+x+1} + C$$

(3) $\displaystyle\int \frac{x}{(x+1)(x^2+1)}\,dx$

【解答】$\displaystyle\frac{x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$ とおいて

$a=-\dfrac{1}{2},\ b=\dfrac{1}{2},\ c=\dfrac{1}{2}$

$$\frac{x}{(x+1)(x^2+1)} = -\frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)}$$

$$\int \frac{x}{(x+1)(x^2+1)}\,dx = -\frac{1}{2}\int \frac{1}{x+1} + \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

2 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{dx}{(x+a)(x+b)}\ (a \neq b)$

【解答】

(1)

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)$$ であるから

$$\int \frac{dx}{(x+a)(x+b)} = \int \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$$

$$= \frac{1}{b-a}\int \left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$$

$$= \frac{1}{b-a}\left(\log|x+a| - \log|x+b|\right) + C$$

$$= \frac{1}{b-a}\log\left|\frac{x+a}{x+b}\right| + C$$

(2) $\displaystyle\int \frac{dx}{x^2-a^2}$

【解答】

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$ であるから

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left(\log|x-a| - \log|x+a|\right) + C$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(3) $\displaystyle\int \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx\ (a \neq 0,\ b \neq 0)$

【解答】

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right)$$ であるから

$$\int \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{b^2-a^2}\int \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right)dx$$

$$= \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$$

有理関数の積分 (部分分数分解)

$\boxed{3}$ 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx$

【解答】基本事項 $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\ (a \fallingdotseq 0)$

$$\int \frac{l\,x+m}{x^2+2px+q}\,dx = \frac{l}{2}\log(x^2+2px+q) + \frac{m-lp}{\sqrt{q-p^2}}\tan^{-1}\frac{x+p}{\sqrt{q-p^2}} + C\ (q>p^2)$$

$\displaystyle\frac{x}{x^4+x^2+1} = \frac{x}{(x^2+x+1)(x^2-x+1)} = \frac{ax+b}{x^2+x+1} + \frac{cx+d}{x^2-x+1}$ とおいて

$$a=0,\ b=-\frac{1}{2},\ c=0,\ d=\frac{1}{2}$$

$$\int \frac{x}{x^4+x^2+1}\,dx = \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{2x-1}{\sqrt{3}} - \tan^{-1}\frac{2x+1}{\sqrt{3}}\right) + C = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2x^2+1} + C$$

(2) $\displaystyle\int \frac{1-x^2}{x(x^4+x^2+1)}\,dx$

【解答】 $\displaystyle\frac{1-x^2}{x(x^4+x^2+1)} = \frac{1}{x} - \frac{x-1}{2(x^2+x+1)} - \frac{x+1}{2(x^2-x+1)}$ として計算できる。

しかし，次のようにするほうが簡単である。

$\displaystyle\frac{1}{x^2} = t$ とおくと $-2\log|x| = \log t$ 両辺を微分して $\displaystyle\frac{dx}{x} = -\frac{dt}{2t}$ を得る。

$$\int \frac{1-x^2}{x(x^4+x^2+1)}\,dx = \int \frac{1-x^2}{x^4+x^2+1}\cdot\frac{dx}{x}$$

$$= \int \frac{1-t^{-1}}{t^{-2}+t^{-1}+1}\left(-\frac{dt}{t}\right) = -\frac{1}{2}\int \frac{t-1}{t^2+t+1}\,dt$$

$$= -\frac{1}{4}\int \frac{2t+1}{t^2+t+1}\,dt + \frac{3}{4}\int \frac{dt}{t^2+t+1}$$

$$= -\frac{1}{4}\log(t^2+t+1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2t+1}{\sqrt{3}} + C$$

$$= \frac{1}{4}\log\frac{x^4}{x^2+x+1} + \frac{\sqrt{3}}{2}\tan^{-1}\frac{x^2+2}{\sqrt{3}x^2} + C$$

(3) $\displaystyle\int \frac{dx}{x(a+bx^n)^2}$

【解答】$a+bx^n = t$ とおけば $nbx^{n-1}\,dx = dt$ したがって $\displaystyle\frac{dx}{x} = \frac{dt}{nbx^n} = \frac{1}{n}\cdot\frac{dt}{t-a}$ を代入して

$$\int \frac{dx}{x(a+bx^n)^2} = \frac{1}{n}\int \frac{dt}{t^2(t-a)} = \frac{1}{na^2}\int\left(\frac{1}{t-a} - \frac{1}{t} - \frac{a}{t^2}\right)dt$$

$$= \frac{1}{na^2}\left(\log\left|\frac{bx^n}{a+bx^n}\right| + \frac{a}{a+bx^n}\right) + C$$

(4) $\displaystyle\int \frac{dx}{x^3+1}$

【解答】 $\displaystyle\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$ とおけば

$$a=\frac{1}{3},\ b=-\frac{1}{3},\ c=\frac{2}{3}$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$ ☜ここまで

$$= \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$

$$= \frac{1}{6}\log\frac{(x+1)^3}{x^3+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{6}\log|x^3+1| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$

(5) $\displaystyle\int \frac{dx}{x^4+1}$

【解答】 $x^4+1 = (x^2+1)^2 - 2x^2 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ より

$\displaystyle\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$ とおくと $\displaystyle a=\frac{1}{2\sqrt{2}},\ b=\frac{1}{2},\ c=-\frac{1}{2\sqrt{2}},\ d=\frac{1}{2}$

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} = \frac{x+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1} = \frac{1}{2}\cdot\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{\sqrt{2}}\cdot\frac{1}{x^2+\sqrt{2}x+1}$$

$$\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} = \frac{x-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1} = \frac{1}{2}\cdot\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} - \frac{1}{\sqrt{2}}\cdot\frac{1}{x^2-\sqrt{2}x+1}$$ より

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left\{\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\right\} + C$$

$$= \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2} + C$$

(注) $\tan^{-1}m_1 = \theta_1,\ \tan^{-1}m_2 = \theta_2$ のとき，$\tan\theta_1 = m_1,\ \tan\theta_2 = m_2$

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} = \frac{m_1+m_2}{1-m_1m_2}$$

$$\therefore\ \theta_1+\theta_2 = \tan^{-1}\left(\frac{m_1+m_2}{1-m_1m_2}\right)$$

$$\therefore\ \tan^{-1}m_1 + \tan^{-1}m_2 = \tan^{-1}\left(\frac{m_1+m_2}{1-m_1m_2}\right)$$

同様に，$\displaystyle\tan^{-1}m_1 - \tan^{-1}m_2 = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$

$$\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1) = \tan^{-1}\frac{(\sqrt{2}x+1)+(\sqrt{2}x-1)}{1-(\sqrt{2}x+1)(\sqrt{2}x-1)} = \tan^{-1}\frac{\sqrt{2}x}{1-x^2}$$

有理関数の積分 (部分分数分解)

4 次の不定積分を求めよ。

(1) $\displaystyle\int \frac{lx+(m-lp)}{x^2+a^2}\,dx$

【解答】

$$\int \frac{lx+(m-lp)}{x^2+a^2}\,dx = \int \frac{lx}{x^2+a^2}\,dx + (m-lp)\int \frac{dx}{x^2+a^2}$$

ここで, $\displaystyle\int \frac{lx}{x^2+a^2}\,dx = l\int \frac{x}{x^2+a^2}\,dx = \frac{l}{2}\log(x^2+a^2)+C_1$

$$(m-lp)\int \frac{dx}{x^2+a^2} = (m-lp)\int \frac{dx}{a^2\left\{\left(\frac{x}{a}\right)^2+1\right\}}$$

$$= \frac{m-lp}{a}\tan^{-1}\frac{x}{a}+C_2$$

$$\therefore \int \frac{lx+(m-lp)}{x^2+a^2}\,dx = \frac{l}{2}\log(x^2+a^2)+\frac{m-lp}{a}\tan^{-1}\frac{x}{a}+C$$

(2) $\displaystyle I=\int \frac{dx}{ax^2+2bx+c}$ $(a \fallingdotseq 0)$

【解答】基本事項 $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$ $(a \fallingdotseq 0)$

$\displaystyle I = a\int \frac{dx}{(ax+b)^2+(ac-b^2)}$ で

$ax+b=t$ とおけば $dx=\dfrac{dt}{a}$ だから

$\displaystyle I=\int \frac{dx}{t^2+(ac-b^2)}$ である。

$ac-b^2>0$ のときは

$$I=\frac{1}{\sqrt{ac-b^2}}\tan^{-1}\frac{t}{\sqrt{ac-b^2}}+C$$

$$=\frac{1}{\sqrt{ac-b^2}}\tan^{-1}\frac{ax+b}{\sqrt{ac-b^2}}+C$$

$ac-b^2=0$ ならば

$$I=-\frac{1}{t}+C=-\frac{1}{ax+b}+C$$

$ac-b^2<0$ ならば

$$I=\frac{1}{2\sqrt{b^2-ac}}\log\left|\frac{t-\sqrt{b^2-ac}}{t+\sqrt{b^2-ac}}\right|+C$$

$$=\frac{1}{2\sqrt{b^2-ac}}\log\left|\frac{ax+b-\sqrt{b^2-ac}}{ax+b+\sqrt{b^2-ac}}\right|+C$$

(3) $\displaystyle I=\int \frac{Lx+M}{x^2+2px^2+q}\,dx$ $(q-p^2>0)$

【解答】

$x^2+2px^2+q=(x+p)^2+q-p^2$ だから $x+p=z,\ q-p^2=a^2$ とおけば

$Lx+M=L(x+p)+(M-Lp),\ dx=dz$ だから

$$I=\int \frac{Lz+(M-Lp)}{z^2+a^2}\,dx$$

$$=\frac{L}{2}\log(z^2+a^2)+\frac{M-Lp}{a}\tan^{-1}\frac{z}{a}+C$$

$$=\frac{L}{2}\log(x^2+2px^2+q)+\frac{M-Lp}{\sqrt{q-p^2}}\tan^{-1}\frac{x+p}{\sqrt{q-p^2}}+C$$