$\triangle$ABC における等式の証明 No.1

$\triangle$ABC において，次の等式が成り立つことを証明せよ。

(1) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(3) $\dfrac{\tan B}{\tan A + \tan B + \tan C} + \dfrac{2\sin 2B}{\sin 2A + \sin 2B + \sin 2C} = 1$

【証明】

(1) $A + B + C = \pi$ より $A + B = \pi - C$

$$\tan(A+B) = \tan(\pi - C)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\therefore\ \tan A + \tan B + \tan C = \tan A \tan B \tan C$

(2) $A + B + C = \pi$ より $C = \pi - (A+B)$

$$\begin{aligned}
\sin 2A + \sin 2B + \sin 2C &= (\sin 2A + \sin 2B) + \sin 2\{\pi - (A+B)\} \\
&= 2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2} - \sin 2(A+B) \\
&= 2\sin(A+B)\cos(A-B) - 2\sin(A+B)\cos(A+B) \\
&= 2\sin(A+B)\{\cos(A-B) - \cos(A+B)\} \\
&= 2\sin(\pi - C)\left\{-2\sin\frac{(A-B)+(A+B)}{2}\sin\frac{(A-B)-(A+B)}{2}\right\} \\
&= -4\sin C \sin A \sin(-B) = 4\sin A \sin B \sin C
\end{aligned}$$

(3)

$$\begin{aligned}
&\frac{\tan B}{\tan A + \tan B + \tan C} + \frac{2\sin 2B}{\sin 2A + \sin 2B + 2\sin 2C} \\
&= \frac{\tan B}{\tan A \tan B \tan C} + \frac{4\sin B \cos B}{4\sin A \sin B \sin C} \\
&= \frac{1}{\tan A \tan C} + \frac{\cos B}{\sin A \sin C} \\
&= \frac{\cos A \cos C}{\sin A \sin C} + \frac{\cos B}{\sin A \sin C} \\
&= \frac{\cos A \cos C + \cos B}{\sin A \sin C} \\
&= \frac{\cos A \cos C - \cos(A+C)}{\sin A \sin C} \\
&= \frac{\cos A \cos C - (\cos A \cos C - \sin A \sin C)}{\sin A \sin C} \\
&= \frac{\sin A \sin C}{\sin A \sin C} = 1
\end{aligned}$$

△ABC における等式の証明 No.2

△ABC において，次の等式が成り立つことを証明せよ。

(1) $\sin A+\sin B+\sin C=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(2) $\cos A+\cos B+\cos C=1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(3) $\cos A+\cos B-\cos C=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}-1$

(4) $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

【証明】

(1) $A+B+C=\pi$ より $C=\pi-(A+B)$

$$\begin{aligned}\sin A+\sin B+\sin C&=(\sin A+\sin B)+\sin\{\pi-(A+B)\}\\&=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-\sin(A+B)\\&=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\\&=2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)\\&=2\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)\cdot2\cos\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\cos\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\\&=4\cos\frac{C}{2}\cos\frac{A}{2}\cos\left(-\frac{B}{2}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\end{aligned}$$

(2) $\cos A+\cos B=2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$$\cos C=\cos\{\pi-(A+B)\}=-\cos(A+B)=-\left(2\cos^2\frac{A+B}{2}-1\right)$$

$$\begin{aligned}\cos A+\cos B+\cos C&=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-\left(2\cos^2\frac{A+B}{2}-1\right)\\&=1+2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\\&=1+2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\left\{-2\sin\frac{A}{2}\sin\left(-\frac{B}{2}\right)\right\}=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\end{aligned}$$

(3) $A+B+C=\pi$ より $C=\pi-(A+B)$

$$\begin{aligned}\cos A+\cos B-\cos C&=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\left(2\cos^2\frac{A+B}{2}-1\right)\\&=2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)-1\\&=2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cdot2\cos\frac{A}{2}\cos\left(-\frac{B}{2}\right)-1=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-1\end{aligned}$$

(4) 半角の公式から $\cos^2 A=\dfrac{1+\cos 2A}{2}$，$\cos^2 B=\dfrac{1+\cos 2B}{2}$

よって $\cos^2 A+\cos^2 B=1+\dfrac{1}{2}(\cos 2A+\cos 2B)=1+\cos(A+B)\cos(A-B)$

また，$C=\pi-(A+B)$ であるから $\cos^2 C=\cos^2\{\pi-(A+B)\}=\cos^2(A+B)$

$$\begin{aligned}\cos^2 A+\cos^2 B+\cos^2 C&=1+\cos(A+B)\cos(A-B)+\cos^2(A+B)\\&=1+\cos(A+B)\cos(A-B)+\cos^2(A+B)\\&=1+\cos(A+B)\left(\cos(A-B)+\cos(A+B)\right)\\&=1+\cos(\pi-C)\cdot2\cos A cos B\\&=1-2\cos A cos B\cos C\end{aligned}$$